Appendix

The missile intercept problem

1) BP flight time to booster is

$$t_F = t_{BO} - t_0 .$$

2) Consider the *vertical* flight of the BP. At what elevation angle must the BP be fired to intercept booster at its burnout point? Intercept occurs when two objects arrive at the same place at the same time (**Figure 5**). To arrive at h_{BO} at time t_{BO} the BP must fly downward at the following speed:

$$V_v = (h_{BO} - h_0)/t_F .$$

This neglects BP acceleration due to firing of its engine. That is, it assumes the acceleration time is short.

If we now assume the earth is flat and neglect gravity effects (the resulting errors are small if the BP flight time is much less than the period of the BP orbit: since flight times are about 5 minutes, and the BP orbital period is about 90 minutes, the assumption is a good one), the vertical component of the BP's *total* speed is given by

$$V_v = \Delta V \sin E.$$

Thus we have

$$-1 \leq \sin E = (h_{BO} - h_0)/\Delta V t_F \leq 1$$

and

$$\cos E = \sqrt{1 - \sin^2 E}.$$

Now, t_F spans a range of values, depending on when the BP is fired relative to booster launch, that is, the value of t_0 . The largest possible value of t_0 for an intercept is given by

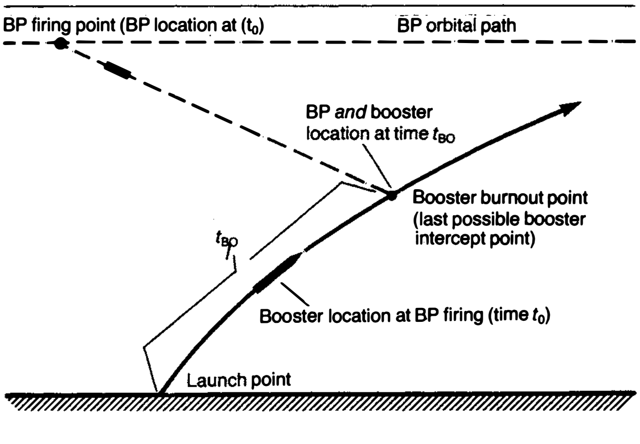
$$\frac{h_0 - h_{BO}}{\Delta V\,(t_{BO} - t_{0_{max}})} = 1.$$

Therefore

$$t_{0_{max}} = t_{BO} - (h_0 - h_{BO})/\Delta V.$$

If $t_0 > t_{0_{max}}$, intercepts are not possible. The minimum possible value of t_0 is zero. *Corresponding to* $t_{0_{min}}$ *and* $t_{0_{max}}$ *are a minimum and maximum value of* E *for which intercepts are possible.*

FIGURE 5
BP intercepting a booster



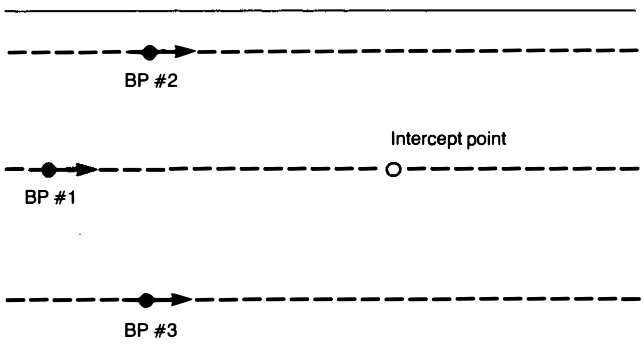
All times measured relative to booster launch.

3) Consider the *horizontal* flight of the BP. At what azimuth angle must the BP be fired to intercept the booster? Assuming a BP's orbit passes directly over the booster intercept point, the horizontal speed of the BP fired at the intercept point is

$$V_H = V_0 + \Delta V \cos E.$$

But what about the situation in which the BP's orbit is to the "left" or "right" of the intercept point (see **Figure 6**)? BPs #2 and #3 in Figure 6 can also fire their engines and reach the intercept point.

FIGURE 6



BPs #2 and #3 can also fire their engines and reach the intercept point.

We define an (x,y,z) coordinate system with origin at the intercept point. Let the z axis be "up" and the x axis be parallel to the BP orbital plane (see **Figure 7**). The BP must fire at some relative azimuth α, as shown in Figure 7, to reach the intercept point.

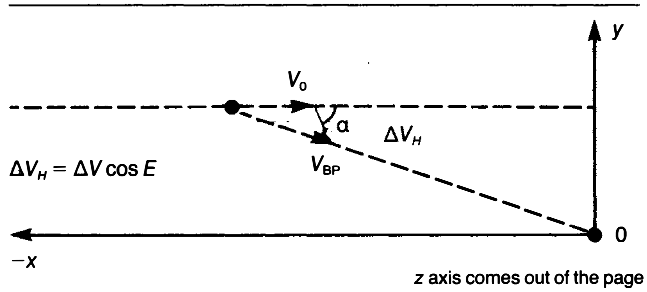
The equations for the BP's velocity in terms of the three components "down," "forward," and "left or right" are

"forward": $V_x = V_0 + \Delta V \cos E \cos \alpha$

"left or right": $V_y = \Delta V \cos E \sin \alpha$

"down": $V_z = \Delta V \sin E$ (equivalent to V_v)

FIGURE 7
The coordinate system



An (x,y,z) coordinate system with origin at the intercept point. The z axis is "up" and the x axis is parallel to the BP orbital plane. The BP must fire at some relative azimuth α to reach the intercept point.

where E=elevation angle from BP's local horizontal, up being positive, and

α=azimuth angle from BPs local vertical, clockwise being positive.

The total absolute speed of the BP is given by

$$V_{BP} = \sqrt{V_x^2 + V_y^2 + V_z^2}.$$

Note that V_x and V_y
is the maximum value of V_x, as well as being the maximum value of the "vector sum" of V_x and V_y,

$$V_H \geq \sqrt{V_x^2 + V_y^2}.$$

The angle α can be solved for in much the same way that we solved for E. For an intercept, we must have

$$V_y = y/t_F \text{ and } V_x = x/t_F .$$

Therefore

$$\sin \alpha = \frac{y}{\Delta V t_F \cos E} \quad \text{and} \quad \cos \alpha = \frac{(x/t_F) - V_0}{\Delta V \cos E} .$$

Of course, the above equations are subject to the restrictions that

$$-1 \leq \frac{y}{\Delta V t_F \cos E} \leq 1$$

and

$$-1 \leq \frac{(x/t_F) - V_0}{\Delta V \cos E} \leq 1 .$$

These restrictions impose limits on x and y for given values of ΔV, t_F, E, and V_0.

Given a y, ΔV,
azimuth is required and how far downrange (value of x) the BP must be to yield an intercept.

It is extremely useful to solve for x and y in terms of the other parameters. Two forms of the solution are possible:

$$\tan \alpha = \sin \alpha / \cos \alpha = y/(x - V_0\, t_F).$$

Therefore

$$y = (x - V_0\, t_F) \tan \alpha, \quad 0 \leq \alpha \leq \alpha_{max} .$$

Rather than the above parametric form, an explicit form

Notation

h_{BO}	=	Altitude of booster burnout point
t_{BO}	=	Time from booster launch to booster burnout (length of boost-phase portion of trajectory)
t_0	=	Time from booster launch to firing of BP
t_F	=	BP flight time to BP/booster intercept point
h_0	=	Altitude of BP orbit
V_0	=	BP orbital speed
	=	$\sqrt{\mu/(R_e+h_0)}$, μ=gravitational constant $=398601.2$ km^3/sec^2; R_e=radius of earth $=6378$ km
ΔV	=	BP gain in speed due to the firing of its propulsion system
V_{BP}	=	BP total speed
V_x	=	BP in-plane component of total speed after firing
V_y	=	BP cross-plane component of total speed after firing
V_z	=	BP vertical component of total speed after firing
V_H	=	Maximum value of V_x
V_v	=	Same as V_z
E	=	Local elevation angle at which BP is pointed and fired
α	=	Local azimuth angle at which BP is pointed and fired
x	=	Distance from booster intercept point in same direction as BP orbital velocity
y	=	Distance from booster intercept point perpendicular to x in the horizontal plane
z	=	Vertical distance up from booster intercept point
N	=	Total number of satellites in orbit
$N_{S/R}$	=	Number of satellites per orbit ring
N_R	=	Number of orbit rings
N_Δ	=	Number of satellites in a region
$\Delta\nu$	=	Angular spacing between satellites in a ring
$\Delta\phi$	=	Angular spacing between rings where they cross the equator

without α can be found, namely

$$\sin^2\alpha + \cos^2\alpha = \frac{y^2}{(\Delta V t_F \cos E)^2} + \frac{(x-V_0\, t_F)^2}{(\Delta V t_F \cos E)^2}\,.$$

Therefore

$$\boxed{y^2+(x-V_0\, t_F)^2=\Delta V^2 t_F^2 - z^2, \qquad z=h_0-h_{BO},}$$

which is the equation for a circle centered at $y=0$, $x=-V_0\, t_F$, and of radius $r=\sqrt{\Delta V^2 t_F^2-z^2}$. Note that we have substituted for cos E from section (2) above. Thus, for t_F a maximum, the above expression defines the largest circle for which a BP on the circle's perimeter can intercept a booster at the booster burnout point. Recall that V_0 is dependent on the BP altitude.

We define the region enclosed by the family of circles corresponding to all feasible values of t_F as the BP boost-phase *firing zone*.

4) Boost-phase battle space. The locus of all regions over the earth containing BPs that can intercept boosters (firing zones) is known as the boost-phase battle space. These are roughly circles (exactly circles for a flat earth) offset from the booster burnout points. The direction of the offset depends on the orbital velocity of the BP, that is, the offset lies on a BP ground track that passes under the booster burnout point.

For BPs in polar orbits (passing over north and south poles) traveling north and south over booster launch complexes, the boost-phase battle space is as shown in **Figure 8**.

5) Number of BPs in boost-phase battle space. If N_B is the number of boosters, then there must be *at least* N_B BPs in the total boost-phase battle space to intercept all of the boosters.

6) Distribution of BPs over the earth. Suppose BPs are placed in polar orbits (orbit passes over North and South poles) and that the spacing of BPs at the equator is uniform at some instant in time (see **Figure 9**). If $N_{S/R}$ is the number of BPs (satellites) per ring, then

$$\Delta\nu=2\pi/N_{S/R}\,.$$

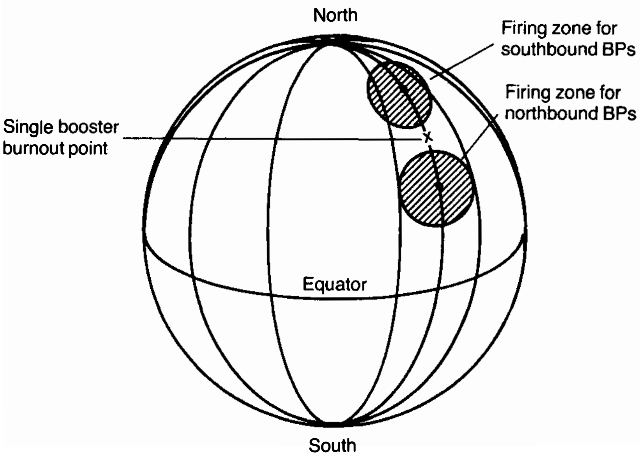
For uniform spacing at the equator (to ensure uniform coverage against submarine launched missiles),

$$\Delta\phi=\Delta\nu,$$

the number of rings is given by

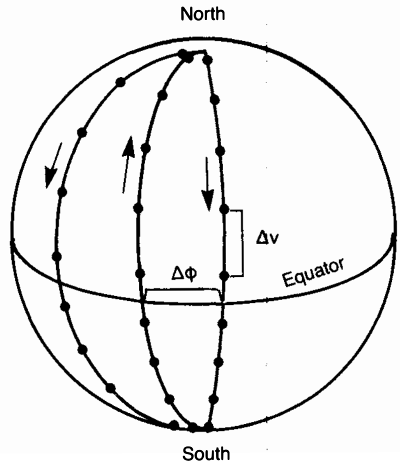
$$N_R=\pi/\Delta\phi,$$

FIGURE 8
The boost-phase battle space



The locus of all firing zones defines the battle space.

FIGURE 9
Distribution of BPs over the earth



To ensure counter-rotating rings, N_R must be an odd number.

and the total number of satellites is given by

$$N=N_{S/R}\,N_R.$$

Suppose a specified region over the earth requires a certain number of BPs to be found within it at every instant, or conversely, given a constellation, suppose we wished to find the number of objects within a specified region. The relevant equation for a uniformly spaced polar constellation is

$$N_\Delta=N(\Delta\text{ longitude}/2\pi)(\Delta\text{ latitude}/\pi)$$

$$=N(\Delta\text{ longitude (deg)}/360)(\Delta\text{ latitude (deg)}/180).$$

If 500 BPs are required to be in a band with Δ latitude$=60°$ and Δ longitude$=90°$, the total number of BPs on orbit, assuming a polar constellation of uniform spacing, is

$$\begin{aligned} N &= N_\Delta\,(360/\Delta\text{ long})(180/\Delta\text{ lat}) \\ &=(500)(4)(3) \\ &=6000. \end{aligned}$$

Note that the *absentee ratio* for the *region* is 1/12.

Roughly half of the BPs in the band are traveling northbound and half are traveling southbound. Thus the band must contain both northbound and southbound BP firing zones if all of the BPs in the band are to be targeted to boosters.

The absentee ratio as defined above is the *instantaneous* absentee ratio, that is, the fraction of BPs within firing zones *at any instant*. If all the boosters are launched simultaneously, the *instantaneous absentee ratio* is equivalent to the *battle absentee ratio*. If boosters are launched over a certain time span, however, new BPs enter the firing zones as boosters are launched. In this case, the battle absentee ratio may be *more or less* than the instantaneous value, but is *less* for typical scenarios.

Suppose, for example, that 1,000 ICBMs are launched over a ten-minute span. Suppose further that BPs enter (and leave) the firing zones at a rate of 100 BPs per minute. (This is easy to calculate given the firing zones and the BP constellation.) For all the boosters to be intercepted, the *rate* of booster launches must be less than the *rate* of BP entry into the firing zones. For our example, therefore,

$$1000\text{ ICBMs}/10\text{ min} \leqslant 100\text{ BPs/min}.$$

Therefore all ICBMs can be targeted.

The *battle* absentee ratio is 1/6 for a 6,000 BP constellation. The sizing of a BP constellation for extended booster launches is straightforward.

Simultaneous launches cause the greatest absentee ratios, and hence this scenario must be used to size the BP constellation if simultaneous launches are a possible threat. The BP constellation gains in utility as the battle is stretched out over time.